International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 5 Issue 1 ǁ January. 2017 ǁ PP. 27-31

Auto Regressive Process (1) with Change Point: Bayesian Approch

Dr. Mayuri Pandya, Paras Sheth
Department of Statistics, Maharaja Krishnkumarsinhji Bhavnagar University,
Bhavnagar, Gujarat, India.

Abstract : Here we consider first order autoregressive process with changing autoregressive coefficient at some point of time m. This is called change point inference problem. For Bayes estimation of m and autoregressive coefficient we used MHRW (Metropolis Hasting Random Walk) algorithm and Gibbs sampling. The effects of prior information on the Bayes estimates are also studied.

Keywords: Auto Regressive Model (1), Bayes Estimator, Change Point, Gibbs Sampling and MHRW Algorithm.

I. INTRODUCTION

Mayuri Pandya (2013) had studied the Bayesian analysis of the autoregressive model $X_t = \beta_1 X_{t-1} + \varepsilon_t$, t=1,2,…,m and $X_t = \beta_2 X_{t-1} + \varepsilon_t$, t=m+1,…,n where $0 < \beta_1, \beta_2 < 1$, and ε_t was independent random variable with an exponential distribution with mean θ_1 and is reflected in the sequence after ε_m is changed in mean θ_2. M. Pandya, K. Bhatt, H. Pandya, C. Thakar (2012) had studied the Bayes estimators of m, β_1 and β_2 under Asymmetric loss functions namely Linex loss & General Entropy loss functions of changing auto regression process with normal error. Tsurumi (1987) and Zacks (1983) are useful references on structural changes.

II. PROPOSED AR (1) MODEL:

Let our AR(1) model be given by,

$$X_i = \begin{cases} \beta_1 X_{i-1} + \epsilon_i, & i = 1,2,\dots,m. \\ \beta_2 X_{i-1} + \epsilon_i, & i = m+1,\dots,n. \end{cases} \quad (1)$$

where, β_1 and β_2 are unknown autocorrelation coefficients, x_i is the i^{th} observation of the dependent variable, the error terms ϵ_i are independent random variables and follow a N(0, σ_1^2) for i=1,2,…,m and a N(0, σ_2^2) for i= m+1, …..,n and σ_1^2 and σ_2^2 both are known. *m* is the unknown change point and x_0 is the initial quantity.

III. BAYES ESTIMATION

The Bayes procedure is based on a posterior density, say, $g(\beta_1, \beta_2, m \mid Z)$, which is proportional to the product of the likelihood function $L(\beta_1, \beta_2, m \mid Z)$, with a joint prior density, say, $g(\beta_1, \beta_2, m)$ representing uncertainty on the parameters values.

The likelihood function of β_1, β_2 and m, given the sample information $Z_t = (x_{t-1}, x_t)$, t = 1, 2,..., m, m+1,..., n. is,

$$L(\beta_1, \beta_2, m|Z) = K_1 \,.\, exp\left(-\frac{1}{2}{\beta_1}^2\left(\frac{S_{m1}}{{\sigma_1}^2}\right) + \beta_1\left(\frac{S_{m2}}{{\sigma_1}^2}\right) - \frac{A_{1m}}{2{\sigma_1}^2}\right) . \,exp\left(-\frac{1}{2}{\beta_2}^2\left(\frac{S_{n1}-S_{m1}}{{\sigma_2}^2}\right) + \beta_2\left(\frac{S_{n2}-S_{m2}}{{\sigma_2}^2}\right) - \frac{A_{2m}}{2{\sigma_2}^2}\right)\sigma_1^{-m}\,\sigma_2^{-(n-m)} \quad (2)$$

Where,

$$S_{k1} = \sum_{i=1}^{k} {x_{i-1}}^2 \qquad S_{k2} = \sum_{i=1}^{k} x_i\, x_{i-1}$$

$$A_{1m} = \sum_{i=1}^{m} {x_i}^2 \qquad A_{2m} = \sum_{i=m+1}^{n} {x_i}^2$$

$$k_1 = (2\pi)^{-\frac{n}{2}} \quad (3)$$

3.1 Using Informative (Normal) Priors On β_1, β_2

In this section, we derive posterior density of change point m, β_1 and β_2 of the model explained in equation (1) under informative priors.

We consider the **AR(1)** model shown in equation (1) with unknown σ^{-2}. We suppose uniform prior of change point same as Broemeling (1987), we also suppose that m, β_1 and β_2 are independent.

$$g(m) = \frac{1}{n-1}$$

We have normal prior density on β_1 and β_2 as,

$$g(\beta_1) = \frac{1}{\sqrt{2\pi}a_1} e^{-\frac{1}{2}\left(\frac{\beta_1}{a_1}\right)^2}$$

$$g(\beta_2) = \frac{1}{\sqrt{2\pi}a_2} e^{-\frac{1}{2}\left(\frac{\beta_2}{a_2}\right)^2}$$

Hence, joint prior p.d.f. of β_1, β_2 and m, say g(β_1, β_2, m) is Joint prior density of β_1, β_2 and m Say $g(\beta_1, \beta_2, m)$ is

$$g(\beta_1, \beta_2, m) = \frac{1}{2\pi a_1 a_2\ (n-1)}\ e^{-\frac{1}{2}\left(\frac{\beta_1}{a_1}\right)^2} e^{-\frac{1}{2}\left(\frac{\beta_2}{a_2}\right)^2} \qquad (4)$$

Using Likelihood function (2) with the joint prior density (4), the joint posterior density of β_1, β_2, m say $g(\beta_1, \beta_2, m|Z)$ is,

$$g(\beta_1, \beta_2, m|Z) = \frac{K_1}{h_1(z)}\left[L(\beta_1, \beta_2, m|Z)\ .\ g(\beta_1, \beta_2, m)\right]$$

$$= \frac{K_2}{h_1(z)}\left[e^{\left[-\frac{1}{2}\beta_1{}^2 A_1 + \beta_1 B_1\right]} e^{\left[-\frac{1}{2}\beta_2{}^2 A_2 + \beta_2\ B_2\right]} e^{\left[-\left(\frac{A_{1m}}{2\sigma_1{}^2} + \frac{A_{2m}}{2\sigma_2{}^2}\right)\right]}\right] \sigma_1^{-m}\ \sigma_2^{-(n-m)} \qquad (5)$$

Where,

$K_2 = \frac{K_1}{2\pi\ a_1 a_2\ (n-1)}$

$A_1 = \frac{S_{m1}}{\sigma_1{}^2} + \frac{1}{a_1{}^2} \qquad B_1 = \frac{S_{m2}}{\sigma_1{}^2}$

$A_2 = \frac{S_{n1}-S_{m1}}{\sigma_2{}^2} + \frac{1}{a_2{}^2} \qquad B_2 = \frac{S_{n2}-S_{m2}}{\sigma_2{}^2} \qquad (6)$

$h_1(Z)$ is the marginal density of z given by,

$h_1(Z) = \sum_{m=1}^{n-1} \int_{\beta_1} \int_{\beta_2} L(\beta_1, \beta_2, m \mid \underline{X})\ .\ \mathrm{g}(\beta_1, \beta_2, m\ ) d\beta_1 d\beta_2$

$$= \sum_{m=1}^{n-1} e^{\left[-\left(\frac{A_{1m}}{2\sigma_1{}^2} + \frac{A_{2m}}{2\sigma_2{}^2}\right)\right]} \sigma_1^{-m}\ \sigma_2^{-(n-m)} \int_{-\infty}^{\infty} e^{\left[-\frac{1}{2}\beta_1{}^2 A_1 + \beta_1 B_1\right]}\ d\beta_1 \int_{-\infty}^{\infty} e^{\left[-\ \frac{1}{2}\beta_2{}^2 A_2 + \beta_2\ B_2\right]}\ d\beta_2$$

$= k_3 \sum_{m=1}^{n-1} T_1(m) \qquad (7)$

Where,

$$T_1(m) = \ k_m G_{1m} G_{2m} \qquad (8)$$

$$G_{1m} = \int_{-\infty}^{\infty} \exp\left[-\ \frac{1}{2}\ \beta_1{}^2 A_1\ +\ \beta_1 B_1\right] d\beta_1 = \frac{e^{\frac{B_1{}^2}{2A_1}}\sqrt{2\pi}}{\sqrt{A_1}} \qquad (9)$$

$$G_{2m} = \int_{-\infty}^{\infty} \exp\left[-\ \frac{1}{2}\ \beta_2{}^2 A_2 +\ \beta_2\ \ B_2\right] d\beta_2 = \frac{e^{\frac{B_2{}^2}{2A_2}}\sqrt{2\pi}}{\sqrt{A_2}} \qquad (10)$$

$$k_m = e^{\left[-\left(\frac{A_{1m}}{2\sigma_1{}^2} + \frac{A_{2m}}{2\sigma_2{}^2}\right)\right]} \sigma_1^{-m}\ \sigma_2^{-(n-m)} \qquad (11)$$

Marginal posterior density of change point m, $\beta_1 and\ \beta_2$ are,

$$g_1(m|x) = \frac{T_1(m)}{\sum_{m=1}^{n-1} T_1(m)} \qquad (12)$$

$$g_1(\beta_1|X) = \frac{k_3}{h_1(X)}\left[\sum_{m=1}^{n-1} k_m e^{\left[-\frac{1}{2}\beta_1{}^2 A_1 + \beta_1 B_1\right]}\right] G_{1m} \qquad (13)$$

$$g_1(\beta_2|X) = \frac{k_3}{h_1(X)}\left[\sum_{m=1}^{n-1} k_m\ e^{\left[-\ \frac{1}{2}\beta_2{}^2 A_2 + \beta_2\ B_2\right]}\right] G_{2m} \qquad (14)$$

G_{1m} , G_{2m} and k_m are define as shows in equation (9), (10) and (11) respectively.

Now, the Bayes estimator of any function of parameter α, say $g(\alpha)$ under the squared loss function is,

$$\boldsymbol{E_{\alpha|z}\big(g(\alpha|Z)\big) = \int_0^{\infty} \alpha\big(g(\alpha|Z)\big)\, d\alpha} \qquad \textbf{(*)}$$

Where, g(α|Z) is marginal posterior density of α. It is complicate to compute equation (*) analytically in this case. Therefore, we use MCMC methods to find the Bayes estimator of β_1, β_2 and m.

Gibbs Sampling algorithm:

Given a posterior distribution g(α|Z) for unknown parameters α=($\alpha_1, \ldots, \alpha_k$) defined, at least up to proportionality, by multiplying the likelihood function with the corresponding prior distribution, we can easily identify the full conditional distribution g(α_i|Z, α_j,j≠i), upto proportionality, by regarding g(α|Z) as a function of α_i(i=1,…,k) only, corresponding all other α_j, j≠i, to be fixed.

To implement the Gibbs sampling procedure, we re-write (13) as full conditional of $\boldsymbol{\beta_1}$, by fixing all other parameters i.e. $\boldsymbol{\beta 2}$ and **m.** Hence full conditional density of $\boldsymbol{\beta_1}$ given $\boldsymbol{\beta_2}$ **and m** is as follows,

$$g(\beta_1 \mid \beta_2, m, Z) \propto N\left(\frac{B_1}{A_1}, \left(\frac{1}{\sqrt{A_1}}\right)^2\right) \qquad (15)$$

where $\mathbf{A_1}$ and $\mathbf{B_1}$ shows in equation (6).

We re-write (14) as full conditional density of □**2** by fixing all other parameters □**1** and **m**, we get the full conditional density of □**2** given □**1**,σ^{-2} and **m** is as follows,

$$g(\beta_2 \mid \beta_1, m, Z) \propto N\left(\frac{B_2}{A_2}, \left(\frac{1}{\sqrt{A_2}}\right)^2\right) \qquad (16)$$

where $\mathbf{A_2}$ and $\mathbf{B_2}$ shows in equation (6).

In order to estimate the parameter β_1, and β_2 we use Gibbs sampling to generate sample from the full conditional density of β_1 and β_2 given respectively in (15) and (16). We use following algorithm:

Algorithm:

Initialize $\beta_1 = \beta_{10}$, $\beta_2 = \beta_{20}$ and $m = m_0$ then,

Step-1: Generate $\beta_1 \sim N\left(\frac{A_1}{B_1}, \left(\frac{1}{\sqrt{B_1}}\right)^2\right)$, using Gibbs Sampling.

Step-2: Generate $\beta_2 \sim N\left(\frac{A_2}{B_2}, \left(\frac{1}{\sqrt{B_2}}\right)^2\right)$, using Gibbs Sampling.

Step-3: Repeat the above steps.

mcmc techniques:

Since the posterior distribution of change point (12) has no closed form, we propose to use MCMC techniques to generate the samples from the posterior distribution. To implement the MCMC Techniques, we re-write (12) as target function of m, by fixing all other parameters i.e. $\boldsymbol{\beta 1}$ and **.** Hence target function of m given $\boldsymbol{\beta 1}$ and is as follows,

$$g(m \mid \beta_1, \beta_2, Z) \propto k_m\, e^{\left[-\frac{1}{2}\beta_1{}^2 A_1 + \beta_1 B_1\right]}\, e^{\left[-\frac{1}{2}\beta_2{}^2 A_2 + \beta_2 B_2\right]} \qquad (17)$$

where $\mathbf{A_1}$, $\mathbf{B_1}$,$\mathbf{A_2}$ $\mathbf{B_2}$ $and\ k_m$ shows in equation (6) and (11) respectively.

IV. NUMERICAL STUDY

Application To Generated Data

Let us consider AR(1) model as

$$X_1 = \begin{cases} 0.1X_{i-1} + \epsilon_i\ , & i = 1, 2, \ldots 10 \\ 0.3\, X_{i-1} + \epsilon_i\ , & i = 11, 12, \ldots 20 \end{cases} \qquad (18)$$

Where, the error terms ϵ_i are independent random variables and follow a $N(0{,}1)$ for i=1,2,…,10. and a $N(0{,}4)$ for i= 11, …..,20 and $\sigma_1^2$and σ_2^2known. m is the unknown change point and $x_0 = 0.1$ is the initial quantity. We have generated 20 random observations from proposed AR(1) model given in (18). The first ten observations are from normal distribution with $\sigma_1{}^2$= 1 and next 10 are from normal distribution with $\sigma_2{}^2$ = 4. β_1 and β_2 themselves were random observations from normal distributions with prior means μ_1 = 0.1, μ_2 = 0.3 and variances a_1 = 0.1 and a_2 =0.1. These observations are given in table-1.

Table -1: Generated observations from proposed AR(1) model.

i	1	2	3	4	5	6	7	8	9	10
X_i	0.167	-0.204	0.399	-0.259	-0.784	-1.058	0.819	0.404	1.215	1.537
ϵ_i	0.157	-0.221	0.420	-0.299	-0.758	-0.979	0.925	0.322	1.175	1.416
i	11	12	13	14	15	16	17	18	19	20
X_i	-3.833	-16.173	9.441	11.857	20.645	1.458	13.249	-9.335	19.812	30.657
ϵ_i	-4.294	-15.023	14.293	9.025	17.088	-4.734	12.812	-13.310	22.613	24.713

To generate a random sample from (9.3) using the RWM-H algorithm, the selected proposal is uniform (2, 19) same as prior, which is symmetric around 10 with small steps. **Since the target function is bounded.** The initial distribution is chosen as uniform (1, 19). Further we truncate the initial distribution and we get integer value of the Bayes estimate of change point (m) is 10 when Selected Proposal is U(1, 19) and Initial

Distribution is U(3, 14). The results are shown in Table 2 for data given in Table1 when given value of β_1=0.1, β_2=0.3, σ_1^2=1 and σ_2^2=16.

Table 2: Bayes Estimates of Change point (m) using RWM-H algorithm under SEL

Bounded	Selected proposal	Initial distribution	Bayes Estimate of Change Point (m)	Integer value of Bayes estimate of Change Point (m)
BD(2,19)	U(1,19)	U(1,19)	8.4	8
BD(2,19)	U(2,19)	U(2,19)	8.6	9
BD(3,19)	U(1,19)	U(1,19)	10.3	10
BD(3,19)	U(1,19)	U(3,14)	10.2	10

We also compute the Bayes estimators of m using RWM-H algorithm for different prior consideration for data given in Table 1. The results are shown in Table 3.

Table 3: Bayes Estimates of Change point (m) using RWM-H algorithm under SEL for different prior consideration.

Sr. No.	a_1^2	a_2^2	Bayes Estimate of change point (m) (Posterior Mean)
1	0.0100	0.01	10
2	0.0400	0.04	10
3	**0.0490**	**0.04**	**10**
4	0.0550	0.09	10
5	0.0600	0.25	10
6	0.0625	0.49	10
7	0.0900	0.64	10
8	0.4900	0.81	10
9	0.8100	1.00	10
10	1.0000	4.00	10

Now we compute the Bayes estimators of β_1 (when given value of β_2=0.3, m=10, σ_1^2 =1 and σ_2^2 =16) and β_2 (when given value of β_1=0.1, m=10, σ_1^2 =1 and σ_2^2 =16) using Gibbs sampling MCMC algorithm for different prior consideration for data given in Table 1. The results are shown in Table 4.

Table 4: Bayes Estimates of β_1 and β_2 using Gibbs Sampling MCMC algorithm under SEL for different prior consideration.

Sr. No.	a_1^2	a_2^2	Bayes Estimates of		S.D. of Bayes Estimates of	
			β_1	β_2	β_1	β_2
1	0.0100	0.01	0.025	0.255	0.048	0.008
2	0.0400	0.04	0.090	0.305	0.048	0.008
3	**0.0490**	**0.04**	**0.107**	**0.305**	**0.048**	**0.008**
4	0.0550	0.09	0.118	0.344	0.048	0.008
5	0.0600	0.25	0.126	0.367	0.048	0.008
6	0.0625	0.49	0.130	0.374	0.048	0.008
7	0.0900	0.64	0.172	0.376	0.048	0.008
8	0.4900	0.81	0.415	0.377	0.048	0.008
9	0.8100	1.00	0.475	0.378	0.048	0.008
10	1.0000	4.00	0.496	0.381	0.048	0.008

Figure 1 graph the full conditional of β_1 when a sample of size 10000 is generated from (9.1), Gibbs Sampling with MCMC algorithm has been run. (β_2=0.3, m=10, σ_1^2 =1 & σ_2^2 =16)

Figure 1: Full Conditional of β_1

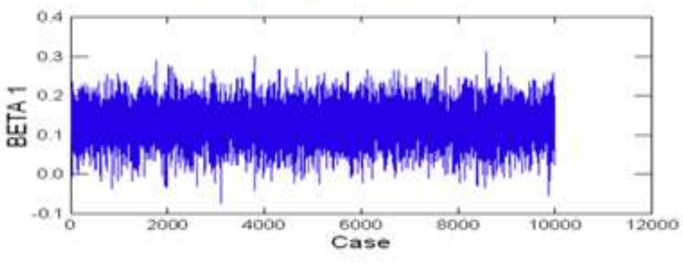
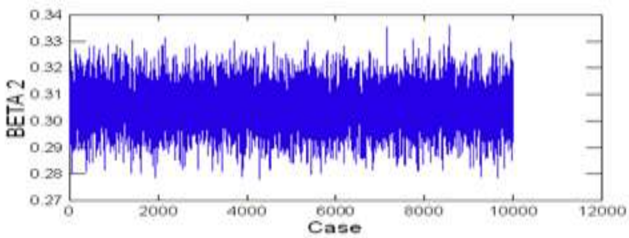


Figure 2 graph the full conditional of β_2 when a sample of size 10000 is generated from (9.2), Gibbs Sampling with MCMC algorithm has been run. (β_1=0.1, m=10, σ_1^2 =1 & σ_2^2 =16)

Figure 2: Full Conditional of β_2



REFERENCES

[1]. Broemeling, L.D. and Tsurumi, H. (1987), Econometrics and structural change, Marcel Dekker, New York.

[2]. Zacks, S. (1983), Survey of classical and Bayesian approaches to the change point problem: fixed sample and sequential procedures for testing and estimation. Recent advances in statistics. Herman Chernoff Best Shrift, Academic Press New-York, 1983, 245-269.

[3]. M. Pandya, K. Bhatt & H. Pandya, C. Thakar (2012), Bayesian estimation of change point in autoregressive process, IJRRAS 13 (1).

[4]. M. Pandya, H. Pandya & S. Pandya (2012), Bayesian inference on mixture of geometric with degenerate distribution: zero inflated geometric distribution, IJRRAS 13 (1).

[5]. M. Pandya & Paras Sheth (2016), Study on Two Phase Linear Regression Model: Bayesian Approach, IJEMR, Volumn-6, Issue-2, page-number: 528-534.